

World-line instantons and the Schwinger effect as a WKB exact path integral

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Abstract

A detailed study of the semiclassical expansion of the world line path integral for a charged relativistic particle in a constant external electric field is presented. We show that the Schwinger formula for charged particle pair production is reproduced exactly by the semiclassical expansion around classical instanton solutions when the leading order of fluctuations is taken into account. We prove that all corrections to this leading approximation vanish and that the WKB approximation to the world line path integral is exact.

1 Introduction

Schwinger's famous formula [1] for what is known as the “Schwinger effect” gives the probability of the production of charged particle-antiparticle pairs by a constant external electric field as

$$P = 1 - e^{-\gamma V} \quad (1)$$

where, for spin zero particles, the exponent is given by

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} E^2}{8\pi^3 n^2} e^{-\pi m^2 n/|E|} \quad (2)$$

Here m is the mass of the particles, V is the space-time volume and E is the electric field. We have absorbed a factor of the particle charge into E .

The result (2) is obtained by evaluating the vacuum persistence amplitude in a theory with a charged massive scalar field exposed to an external electric field.

The phase in the persistence amplitude, which normally contains the vacuum energy, obtains an imaginary part. This gives a damping of the amplitude which is attributed to the production of charged particle-antiparticle pairs. The problem of finding the damping rate can be posed as that of evaluating the imaginary part of the world-line path integral for the relativistic particle,

$$\gamma = -2\Im \frac{1}{V} \int_0^\infty \frac{dT}{T} \int [dx_\mu(\tau)] e^{-\int_0^1 d\tau \left[\frac{T}{4} \dot{x}_\mu(\tau) \dot{x}_\mu(\tau) + E x_1(\tau) \dot{x}_2(\tau) \right] - \frac{m^2}{T}} \quad (3)$$

The integral is over periodic paths, $x_\mu(\tau + 1) = x_\mu(\tau)$ and the space-time metric has Euclidean signature. Of course, a functional integral such as (3) must be defined with care. In this paper, we will use zeta-function regularization in order to define the formally divergent infinite products and infinite summations which are encountered in the course of computing (3). Basic formulae involving zeta functions are summarized in Appendix A. In Appendix B, we shall give a detailed review of the derivation of this path integral formula from the usual Feynman diagram representation of the vacuum persistence amplitude. In particular, we demonstrate that the path integral (3) with $E = 0$ and with zeta function regularization reproduces the Feynman diagram expression for the vacuum energy in all of its details, including its normalization.

The path integral in (3) can be evaluated. The real part can be presented as an integral over one variable and the imaginary part can be found to coincide with (2). The way that it is solved is to first perform the Gaussian integration over the position variables $x_\mu(\tau)$ in (3). In this integration, the instability of the vacuum state of the system of charged particles when a constant electric field is applied is reflected by the subtlety that the quadratic form in the Gaussian functional integral is not positive for all values of T . This is what allows this integral of a real function over real variables to have an imaginary part. The integral is done by first assuming that T is in a region where the Gaussian is stable, doing the Gaussian functional integral over $x_\mu(\tau)$ and then defining the result of the integral for all values of T by analytic continuation. The presence of values of T where the path integral was unstable is then reflected as singularities in the remaining integration variable, T . In this case, the singularities are simple poles on the real T -axis which must be defined carefully to take causal boundary conditions into account. The imaginary part of the integral then comes from the sum over the residues of the poles. This yields the infinite series quoted in (2) above. This is straightforward. It leads to the exponents in the individual terms in (2) and, with some care in normalizing the Gaussian functional integral involved, to the exact pre-factors in (2).

There is another approach to computing the imaginary part of the path integral, which is less efficient, but it is often used as a starting point for computations of the rate of particle production in the more general situation where the electric field is not constant [2]-[3]. It has also been used to discuss pair production in the context of AdS/CFT holography [4]. This approach is a conventional semiclassical evaluation of the path integral. It is generally good when the particle mass is large compared to other dimensionful parameters. In our case, the parameter which controls the semiclassical limit is the dimensionless ratio of the electric field strength to the mass squared, $\frac{E}{m^2}$, which is small in the “weak field limit”. In this limit, we treat both $x_\mu(\tau)$

and T as dynamical variables and solve the classical equations of motion which follow from the world-line action,

$$S = \int_0^1 d\tau \left[\frac{T}{4} \dot{x}_\mu(\tau)^2 + E x_1(\tau) \dot{x}_2(\tau) + \frac{m^2}{T} \right] \quad (4)$$

where $\dot{x}_\mu \equiv \frac{d}{d\tau} x^\mu(\tau)$. The classical solutions, which we denote as $x_{0\mu}(\tau)$ and T_0 , are a saddle point of the path integral integrand. We then compute the integral by saddle point technique which amounts to changing integration variables as

$$T \rightarrow T_0 + \delta T \quad (5)$$

$$x_\mu(\tau) \rightarrow x_{0\mu}(\tau) + \delta x_\mu(\tau) \quad (6)$$

and implementing perturbation theory in the fluctuations δT and $\delta x_\mu(\tau)$. This turns out to be an expansion in the parameter $\sqrt{\frac{E}{m^2}}$ and the expansion is valid in the regime where this parameter is small.

There is a beautiful observation, due to Affleck, Alvarez and Manton [5] that the classical solutions that are relevant to the Schwinger process can be interpreted as instantons. The n 'th term in the summation in the Schwinger formula (2) can be interpreted as a n -instanton amplitude in such a semi-classical computation of the path integral. They showed explicitly that the first, $n = 1$ term in (2),

$$\gamma_1 = \frac{E^2}{8\pi^3} e^{-\pi m^2/|E|} \quad (7)$$

is obtained exactly by such a semi-classical computation where they expand about a one-instanton solution of the classical equations of motion for $x_\mu(\tau)$ and T . The exponent in (7) is the classical action of the instanton. The pre-factor is given by the Gaussian integral over fluctuations about the classical solution at the leading, quadratic order. It is interesting that, in the computation presented by Affleck, Alvarez and Manton, the integral is given exactly by what amounts to the leading orders of an approximation. If it were an approximation, the small parameter which suppresses corrections would be $\sqrt{\frac{E}{m^2}}$. However, given that, in the leading orders they already obtained the exact result, as they noted, but did not demonstrate, higher order perturbative corrections should then cancel exactly. This would mean that the computation has a much larger regime where it is valid, in principle for all values of $\sqrt{\frac{E}{m^2}}$.

The nature of the instanton is easy to understand. In Euclidean space, a Minkowski space electric field behaves as a magnetic field. In a magnetic field, the classical charged particle has a cyclotron orbit. The one-instanton solution is a single cyclotron orbit. The exponent of (7) is simply the classical action of the world line theory evaluated on this orbit. The pre-factor in (7) is given by evaluating the Gaussian integral over the fluctuations about this classical solution.

In this approach, the path integral gets an imaginary part due to the fact that the instantons in question are unstable solutions of the classical world-line theory.

The unstable fluctuation turns out to be the fluctuation of the radius of the cyclotron orbit. The Gaussian integral over the fluctuations, including the fluctuation of the radius, then produces the square root of a determinant of a matrix which has an odd number of negative eigenvalues, thus the factor of “i”.

As well as a single instanton that leads to (7), there are an infinite series of multi-instanton classical solutions which are simply the multiple cyclotron orbits. In the following, we shall show that all of the higher terms in (2),

$$\gamma_n = (-1)^{n+1} \frac{E^2}{8\pi^3 n^2} e^{-\pi m^2 n/|E|} \quad , \quad (8)$$

with $n = 2, 3, \dots$ are produced by multi-instantons with higher wrapping number. It is easy to see (and already well known) that the exponent of the n 'th term as displayed in (8) is the classical action of the n -instanton solution. What we shall show is that the fluctuation integral produces the pre-factor of the exponential exactly. This has the interesting implication that the full, exact result is obtained in the semi-classical Gaussian approximation of the world-line path integral where one sums over all of the classical solutions. Of course, the Gaussian approximation normally has corrections coming from expanding in the higher order non-Gaussian terms in the action, as well as corrections from an expansion about the classical solution of the terms which appear in the integration measure. Such terms are indeed at least formally present in this semiclassical expansion. What we shall find here, that the leading approximation produces the exact result, implies that the corrections must cancel. We shall then give a proof that this is indeed the case: all such corrections vanish. The proof uses a simple scaling argument to localize the path integral on its semiclassical limit. This proof expands the range of validity of the semiclassical computation from the weak field limit to the strong field regime. Whether this can help computations in less ideal problems, for example, where the electric field is not constant, is at this point an open question.

In section 2, we shall perform the semiclassical computation of the path integral in equation (3) in the n -instanton sector. We shall define the infinite products and sums which we encounter using zeta function regularization. We show that, by careful treatment of the functional integration measure, the n 'th term in the Schwinger formula (2), including the exact pre-factor, is obtained.

In section 3, we examine higher order corrections beyond the leading order in the saddle point approximation. We find a proof that all corrections beyond the integration of quadratic fluctuations must vanish. The result is that, for computing the imaginary part of the vacuum persistence amplitude, the semiclassical limit of the world-line path integral with an external electric field is exact.

The definitions and values of the relevant zeta functions are summarized in Appendix A. A proof that the usual quantum scalar field theory vacuum energy derived from the vacuum bubble Feynman diagram is identical to the world-line path integral defined using zeta function regularization is outlined in Appendix B. In Appendix C we demonstrate the semiclassical technique that we use on a simple example.

2 Semiclassical evaluation of the world-line path integral

We shall begin with the case of a spinless charged particle of mass m which is subject to a constant external electric field. Its vacuum energy is given by the world-line path integral (3). The instability of the vacuum to the production of on-shell particle-antiparticle pairs is reflected by the fact that the vacuum energy has an imaginary part. We shall compute this imaginary part in a semi-classical expansion about a classical solution of the world-line theory.

To begin, we shall first solve the classical equations of motion which are obtained by varying the world-line action by the dynamical variables T and $x_\mu(\tau)$,

$$\frac{1}{4m^2} \int_0^1 \dot{x}^2 = \frac{1}{T^2}, \quad -\frac{T}{2} \ddot{x}_1 - E \dot{x}_2 = 0, \quad -\frac{T}{2} \ddot{x}_2 + E \dot{x}_1 = 0, \quad -\frac{T}{2} \ddot{x}_{3,4} = 0 \quad (9)$$

with periodic boundary conditions, $x_\mu(\tau + 1) = x_\mu(\tau)$. The solutions of these equations are

$$x_1 = \frac{m}{E} \cos 2\pi n \tau, \quad x_2 = \frac{m}{E} \sin 2\pi n \tau, \quad x_{3,4} = 0, \quad T = \frac{E}{\pi n} \quad (10)$$

Plugging these solutions into the action (4) gives $S_{\text{cl.}} = \frac{\pi n m^2}{E}$, the same expression which appears in the exponents of the terms in (2).

Now, we define the path integration variables as the classical solutions plus fluctuations,

$$x_1 = \frac{m}{E} \cos 2\pi n \tau + \delta x_1, \quad x_2 = \frac{m}{E} \sin 2\pi n \tau + \delta x_2, \quad x_{3,4} = \delta x_{3,4}, \quad T = \frac{E}{\pi n} + \delta T \quad (11)$$

and we expand the action to quadratic order in the fluctuations. We obtain

$$S = \frac{\pi n m}{E} + \frac{2m^2(\pi n)^3}{E^3} \frac{\delta T^2}{2} + \frac{m}{2E} (2\pi n)^2 \delta T \int d\tau (\cos(2\pi n \tau) \delta x_1 + \sin(2\pi n \tau) \delta x_2) \\ + \frac{E}{4\pi n} \int d\tau [\delta \dot{x}^2 - 4\pi n \delta x_1 \delta \dot{x}_2] + \dots \quad (12)$$

To proceed, we shall use the mode expansion

$$\delta x_\mu(\tau) = x_\mu + \sum_{k=1}^{\infty} \left[\sqrt{2} \cos(2\pi k \tau) a_{k\mu} + \sqrt{2} \sin(2\pi k \tau) b_{k\mu} \right] \quad (13)$$

We first note that the action will not depend of the constant modes x_μ . These are space-time translation zero modes. Their integration will result in the overall factor of the space-time volume V in front of the functional integral.

When we substitute (13) into (12), the action becomes

$$\begin{aligned}
S = & \frac{\pi n m^2}{E} + \frac{2m^2(\pi n)^3}{E^3} \frac{\delta T^2}{2} \\
& + \frac{m(2\pi n)^2}{2E} \delta T \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) + \frac{4\pi n E}{2} \left(\frac{a_{n1} - b_{n2}}{\sqrt{2}} \right)^2 + \frac{4\pi n E}{2} \left(\frac{a_{n2} + b_{n1}}{\sqrt{2}} \right)^2 \\
& + \frac{E}{4\pi n} \sum_{k=1, \neq n}^{\infty} (2\pi k)^2 \left[(a_{k\mu}^2 + b_{k\mu}^2) - \frac{2n}{k} (a_{k1} b_{k2} - a_{k2} b_{k1}) \right] \\
& + \frac{1}{4} \delta T \sum_k (2\pi k)^2 [(a_k^\mu)^2 + (b_k^\mu)^2] + \sum_{k=3}^{\infty} m^2 \left(\frac{\pi n}{E} \right)^{k+1} (-\delta T)^k
\end{aligned} \tag{14}$$

The last line of (14) contain terms of higher order than quadratic in the fluctuations. We have written them in this formula for future reference. To the leading order that we are studying in this section, they will be neglected.

In the previous, quadratic terms in equation (14), we have separated the degrees of freedom $(a_{n1,2}, b_{n1,2})$ which have the same frequency as the classical solution and we have written them in the second line. Note that the combination $a_{n2} - b_{n1}$ does not appear in the quadratic terms in the action – this combination is a zero mode. The existence of the zero mode is due to a symmetry, the translation invariance in τ of the action. The integration measure, as we shall define it, is also invariant under translations of τ . The world-line theory is thus τ -translation invariant. However, the instanton solution depends on τ and it is not invariant. The result is a zero mode in the fluctuations about the solution.

The way to handle the presence of a zero mode is by using the Faddeev-Popov trick to introduce a collective variable. This technique effectively substitutes $\delta((a_{n2} - b_{n1})/\sqrt{2})$, accompanied by a Jacobian, into the integrand, and it multiplies the integral by a factor of the volume of the symmetry group, $\int_0^1 d\tau = 1$, in this case.

The introduction of a collective coordinate begins with inserting a factor of one into the path integral using the identity

$$1 = \frac{1}{\omega} \int_0^1 dt \delta(\chi(t)) \left| \frac{d}{dt} \chi(t) \right| \tag{15}$$

where the function $\chi(t)$ should be chosen so that the integration over the zero mode becomes well-defined. Here, ω is the number of solutions of $\chi(t) = 0$ in the interval $t \in [0, 1]$. We shall use the constraint

$$\begin{aligned}
\chi(t) &= \int_0^1 d\tau (\sin(2\pi n \tau) x_1(\tau - t) - \cos(2\pi n \tau) x_2(\tau - t)) \\
&= \frac{1}{\sqrt{2}} \left[\left(\left[\frac{m}{\sqrt{2}E} + a_{n1} \right] \sin(2\pi n t) + b_{n1} \cos(2\pi n t) \right) - \left(a_{n2} \cos(2\pi n t) - \left[\frac{m}{\sqrt{2}E} + b_{n2} \right] \sin(2\pi n t) \right) \right]
\end{aligned} \tag{16}$$

We shall later set $t = 0$ by translating the time variable in the path integral. The constraint reduces to $\chi(0) = \frac{1}{\sqrt{2}} [b_{n1} - a_{n2}]$ which is what we need to constrain the zero mode.

As a function of t , $\chi(t) = 0$ when

$$\tan(2\pi nt) = \frac{a_{n2} - b_{n1}}{\frac{m}{\sqrt{2}E} + a_{n1} + \frac{m}{\sqrt{2}E} + b_{n2}}$$

This equation is periodic in t and it traverses $2n$ periods as t varies from zero to one. The fundamental domain (where it traverses one period) can be taken as $-\frac{1}{4n} < t < \frac{1}{4n}$ and has length $1/2n$. This fixes the constant in (15) as $\omega = 2n$.

The Jacobian evaluated on the constraint is

$$\frac{1}{\omega} \left| \frac{d}{dt} \chi(t) \right|_{\chi=0} = \frac{2\pi n}{\omega} \int_0^1 d\tau [\cos(2\pi n\tau)x_1(\tau) + \sin(2\pi n\tau)x_2(\tau)] = \left| \pi \frac{m}{E} + \pi \frac{a_{1n} + b_{2n}}{\sqrt{2}} \right| \quad (17)$$

The net effect of this procedure is the insertion of the delta function and measure factor

$$\delta \left(\frac{a_{n2} - b_{n1}}{\sqrt{2}} \right) \left[\pi \frac{m}{E} + \pi \frac{a_{1n} + b_{2n}}{\sqrt{2}} \right] \quad (18)$$

into the functional integral. This suppresses the integration over the zero mode and, in the leading order where we keep only the classical part of the Jacobian, it inserts the factor

$$\pi \frac{m}{E} \quad (19)$$

into the measure.

We are now prepared to do the Gaussian integral. The integration of the variables

$$\left[\delta T, \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right), \left(\frac{a_{n1} - b_{n2}}{\sqrt{2}} \right), \left(\frac{a_{n2} + b_{n1}}{\sqrt{2}} \right) \right]$$

gives the measure factor

$$(2\pi)^2 \det^{-\frac{1}{2}} \begin{bmatrix} \frac{2m^2(\pi n)^3}{E^3} & \frac{m(2\pi n)^2}{2E} & 0 & 0 \\ \frac{m(2\pi n)^2}{2E} & 0 & 0 & 0 \\ 0 & 0 & 4\pi n E & 0 \\ 0 & 0 & 0 & 4\pi n E \end{bmatrix} = \pm i (2\pi)^2 \frac{2E}{m(2\pi n)^2} \frac{1}{4\pi n E} \quad (20)$$

where the factor of i arises from the fact that the determinant is negative, and the plus or minus reflects the fact that there is a choice of sign when the square root is taken.

Then, we can integrate over all of the other modes. The result is the infinite product

$$\begin{aligned} & \prod_{k=1}^{\infty} \left(2\pi \frac{2\pi n}{E} \right)^2 (2\pi k)^{-4} \prod_{k=1, \neq n}^{\infty} \left(2\pi \frac{2\pi n}{E} \right)^2 (2\pi k)^{-4} \left(\det \begin{bmatrix} 1 & 0 & 0 & -\frac{n}{k} \\ 0 & 1 & \frac{n}{k} & 0 \\ 0 & \frac{n}{k} & 1 & 0 \\ -\frac{n}{k} & 0 & 0 & 1 \end{bmatrix} \right)^{-\frac{1}{2}} \\ & = \left(2\pi \frac{2\pi n}{E} \right)^{4\zeta(0)-2} (2\pi n)^4 \prod_{k=1, \neq n}^{\infty} \frac{1}{1 - \frac{n^2}{k^2}} = \frac{E^4}{16\pi^4} \prod_{k=1, \neq n}^{\infty} \frac{1}{1 - \frac{n^2}{k^2}} \quad (21) \end{aligned}$$

In the above formula and in the following, we define infinite products using zeta function regularization. Some of the conventions and the zeta functions that are needed are reviewed in Appendix A.

We find the identity

$$\prod_{k=1, \neq n}^{\infty} \frac{1}{1 - \frac{n^2}{k^2}} = \lim_{\alpha \rightarrow n} \prod_{k=1}^{\infty} \frac{1 - \frac{\alpha^2}{n^2}}{1 - \frac{\alpha^2}{k^2}} = \lim_{\alpha \rightarrow n} \frac{\pi \alpha (1 - \frac{\alpha^2}{n^2})}{\sin \pi \alpha} = 2(-1)^{n+1} \quad (22)$$

Gathering measure factors (19), (20) and (22),

- The factor of $\frac{1}{T}$ in the integrand

$$\pi \frac{m}{E}$$

- The Faddeev-Popov determinant

$$\frac{\pi n}{E}$$

- The integral over the tachyon

$$(\pm)i(2\pi)^2 \frac{2E}{m(2\pi n)^2} \frac{1}{4\pi n E}$$

- The integral over all other modes

$$\frac{E^4}{16\pi^4} \cdot 2(-1)^{n+1}$$

we get the result

$$\pi \frac{m}{E} \cdot \frac{\pi n}{E} \cdot (\pm)i(2\pi)^2 \frac{2E}{m(2\pi n)^2} \frac{1}{4\pi n E} \cdot \frac{E^4}{16\pi^4} \cdot 2(-1)^{n+1} = \pm i \frac{E^2}{16\pi^3 n^2} (-1)^{n+1} \quad (23)$$

With the appropriate choice of sign, and the factor of 2 from the formula (3), we can see that we obtain, as the pre-factor of the exponential of the classical action, the factor $\frac{E^2}{8\pi^3 n^2} (-1)^{n+1}$ which matches the pre-factors of the exponential in each term in the summation (2) exactly. The semiclassical integration has given us the exact result for the imaginary part of the integral in the n -instanton sector, $\gamma_n = (-1)^{n+1} \frac{E^2}{8\pi^3 n^2} e^{-\pi m^2 n/|E|}$. Summation of the instanton number results in the sum over n which appears in the Schwinger formula.

It is interesting that we have produced the imaginary part of the functional integral exactly at this order of what is putatively an approximate computation. This means that all of the higher order corrections to this approximation must cancel. We shall explore this issue in the next section.

3 No more corrections

Now let us examine the corrections to the saddle point approximation which we performed in the previous section. Corrections arise from the expansion of the integrand about the saddle-point. If we expand the non-Gaussian parts of the integrand in a power series in the fluctuations, we can use the functional version of Wick's theorem to compute the corrections. Since we have already obtained the exact result in the next-to-leading order of this expansion, we expect that the higher order corrections must find a way to vanish. In this Section, we shall prove that they indeed vanish.

We consider the action (14) and we make the change of variables

$$x_\mu(\tau) = \tilde{x}_\mu(\tilde{\tau}) \quad , \quad \tilde{\tau} = \tau\beta \quad , \quad \tilde{x}_\mu(\tilde{\tau}) = \tilde{x}_\mu(\tilde{\tau} + \beta) \quad (24)$$

$$T = \tilde{T}/\sqrt{\beta} \quad (25)$$

The path integral measure $[dx_\mu]$ is invariant under this change of variables.¹ The scaling of T cancels in the measure of the integral. The world-line action becomes (dropping the tildes)

$$S = \int_0^\beta d\tau \left[\sqrt{\beta} \frac{T}{4} \dot{x}^\mu(\tau)^2 + Ex^0(\tau) \dot{x}^1(\tau) \right] + \sqrt{\beta} \frac{m^2}{T} \quad (26)$$

The path integral cannot depend on the parameter β . Moreover the limit where β is large is the semiclassical limit. In the following we shall take this limit with some care to show that it indeed projects the full path integral to the semiclassical one which we computed in the previous section, where we only kept the classical and Gaussian terms in the action and the classical terms in the integration measure. Again, we shall expand the integration variables about the classical solution,

$$\begin{aligned} x_1(\tau) &= \sqrt{\beta} \frac{m}{E} \cos \frac{2\pi n \tau}{\beta} + \delta x_1(\tau) \quad , \quad x_2(\tau) = \sqrt{\beta} \frac{m}{E} \sin \frac{2\pi n \tau}{\beta} + \delta x_2(\tau) \\ x_{3,4}(\tau) &= \delta x_{3,4}(\tau) \quad , \quad T = \sqrt{\beta} \frac{E}{\pi n} + \delta T \end{aligned} \quad (27)$$

¹To demonstrate this, we can show that the following Gaussian integral is independent of β :

$$\int [dx_\mu] e^{-\int_0^\beta d\tau \frac{T}{4} \beta \dot{x}_\mu^2} \quad , \quad x_\mu(\tau + \beta) = x_\mu(\tau)$$

Using the mode expansion (28) and $[dx_\mu] = dx_\mu \prod_{n=1}^\infty da_{n\mu} db_{n\mu}$,

$$\begin{aligned} \int [dx_\mu] e^{-\int_0^\beta d\tau \frac{T}{4} \beta \dot{x}_\mu^2} &= \int dx_\mu V \delta^D \left(\frac{x_\mu}{\sqrt{\beta}} \right) \prod_{n=1}^\infty da_{n\mu} db_{n\mu} \exp \left(-\frac{T}{4\beta} \sum_{n=1}^\infty (2\pi n)^2 (a_{n\mu}^2 + b_{n\mu}^2) \right) \\ &= V \beta^{D/2} \prod_{n=1}^\infty \left[\frac{4\pi\beta}{(2\pi n)^2 T} \right]^D = V \left[\frac{T}{4\pi} \right]^{D/2} \end{aligned}$$

where we have gauge fixed by inserting $1 = \int dX^\mu \delta^D \left(X^\mu - \frac{1}{\beta} \int_0^\beta d\tau x^\mu(\tau) \right)$. This results in the factor $V \delta^D \left(\frac{x_\mu}{\sqrt{\beta}} \right)$ with V the space-time volume. We have also used zeta function regularization to define the infinite product. The result does not depend on β . This is so in any dimension D .

with the fluctuations expanded as

$$\delta x_\mu(\tau) = \frac{x_\mu}{\sqrt{\beta}} + \sum_{k=1}^{\infty} \left[\sqrt{\frac{2}{\beta}} \cos \frac{2\pi k\tau}{\beta} a_{k\mu} + \sqrt{\frac{2}{\beta}} \sin \frac{2\pi k\tau}{\beta} b_{k\mu} \right] \quad (28)$$

The measure in the path integral is now

$$\frac{d\delta T}{\sqrt{\beta} \frac{E}{\pi n} + \delta T} V \delta \left(\frac{x_\mu}{\sqrt{\beta}} \right) \delta \left(\frac{b_{n1} - a_{n2}}{\sqrt{2}} \right) \left[\beta \frac{\pi m}{E} + \pi \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) \right] \prod_{\mu=1}^4 dx_\mu \prod_{k=1}^{\infty} da_{k\mu} db_{k\mu} \quad (29)$$

The action becomes

$$\begin{aligned} S = & \frac{\pi n m^2}{E} + \frac{2m^2(\pi n)^3}{\beta E^3} \frac{\delta T^2}{2} \\ & + \frac{m(2\pi n)^2}{2\beta^{\frac{1}{2}} E} \delta T \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) + \frac{4\pi n E}{2\beta} \left(\frac{a_{n1} - b_{n2}}{\sqrt{2}} \right)^2 + \frac{4\pi n E}{2\beta} \left(\frac{a_{n2} + b_{n1}}{\sqrt{2}} \right)^2 \\ & + \frac{E}{4\pi n \beta} \sum_{k=1, \neq n}^{\infty} (2\pi k)^2 \left[(a_{k\mu}^2 + b_{k\mu}^2) - \frac{2n}{k} (a_{k1} b_{k2} - a_{k2} b_{k1}) \right] \\ & + \frac{1}{4\beta^{\frac{3}{2}}} \delta T \sum_{k=1}^{\infty} (2\pi k)^2 [(a_k^\mu)^2 + (b_k^\mu)^2] + \sum_{k=3}^{\infty} m^2 \frac{1}{\beta^{\frac{k}{2}}} \left(\frac{\pi n}{E} \right)^{k+1} (-\delta T)^k \end{aligned} \quad (30)$$

Now, in order to make the quadratic terms β -independent, we rescale

$$\delta x^\mu(\tau) \rightarrow \sqrt{\beta} \delta x^\mu(\tau) \quad (31)$$

The Jacobian for this transformation is $(\sqrt{\beta})^{4+8\zeta(0)} = 1$ where we have used $\zeta(0) = -1/2$. The integration measure becomes

$$\begin{aligned} & \frac{d\delta T}{\sqrt{\beta} \frac{E}{\pi n} + \delta T} V \delta(x_\mu) \delta \left(\sqrt{\beta} \frac{b_{n1} - a_{n2}}{\sqrt{2}} \right) \left[\beta \frac{\pi m}{E} + \sqrt{\beta} \pi \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) \right] \prod_{\mu=1}^4 dx_\mu \prod_{k=1}^{\infty} da_{k\mu} db_{k\mu} \\ & = \frac{d\delta T}{\frac{E}{\pi n} + \frac{\delta T}{\sqrt{\beta}}} V \delta(x_\mu) \delta \left(\frac{b_{n1} - a_{n2}}{\sqrt{2}} \right) \left[\frac{\pi m}{E} + \frac{\pi}{\sqrt{\beta}} \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) \right] \prod_{\mu=1}^4 dx_\mu \prod_{k=1}^{\infty} da_{k\mu} db_{k\mu} \end{aligned}$$

and the action is

$$\begin{aligned} S = & \frac{\pi n m^2}{E} + \frac{2m^2(\pi n)^3}{\beta E^3} \frac{\delta T^2}{2} \\ & + \frac{m(2\pi n)^2}{2E} \delta T \left(\frac{a_{n1} + b_{n2}}{\sqrt{2}} \right) + \frac{4\pi n E}{2} \left(\frac{a_{n1} - b_{n2}}{\sqrt{2}} \right)^2 + \frac{4\pi n E}{2} \left(\frac{a_{n2} + b_{n1}}{\sqrt{2}} \right)^2 \\ & + \frac{E}{4\pi n} \sum_{k=1, \neq n}^{\infty} (2\pi k)^2 \left[(a_{k\mu}^2 + b_{k\mu}^2) - \frac{2n}{k} (a_{k1} b_{k2} - a_{k2} b_{k1}) \right] \\ & + \frac{1}{4\beta^{\frac{1}{2}}} \delta T \sum_{k=1}^{\infty} (2\pi k)^2 [(a_k^\mu)^2 + (b_k^\mu)^2] + \sum_{k=3}^{\infty} m^2 \frac{1}{\beta^{\frac{k}{2}}} \left(\frac{\pi n}{E} \right)^{k+1} (-\delta T)^k \end{aligned} \quad (32)$$

Now, we remember that the integral is independent of β . The original integral that we computed was for the case $\beta = 1$. Assuming smooth behavior in β , we can set the original integral equal to the limit of the above as $\beta \rightarrow \infty$. In that limit, the interaction terms in the action and in the measure go to zero and the integral is reduced to the Gaussian one which we have already computed in the previous section where we found that it gives the exact result.

A The Riemann zeta function

In this appendix, we review some properties of the zeta function which are needed in the following appendix to derive the world-line path integral and in Section 2 for the definition of infinite products and summations which are encountered in the Gaussian functional integral which is done there. A more thorough review of zeta functions and relevant discussion can be found in many references, for example, reference [6][7].

The Riemann zeta function is defined by the infinite sum

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (33)$$

defined as a function of a complex variable s where the real part of s should be large enough so that the sum converges. The function is then analytically continued to the entire complex plane where it has simple poles.

The values of the zeta function and its derivative which we use are

$$\zeta(0) = \lim_{s \rightarrow 0} \sum_{k=1}^{\infty} \frac{1}{k^s} = -\frac{1}{2} \rightarrow \prod_{k=1}^{\infty} \alpha = \lim_{s \rightarrow 0} \prod_{k=1}^{\infty} \alpha^{\frac{1}{k^s}} = \alpha^{\zeta(0)} = \alpha^{-\frac{1}{2}} \quad (34)$$

and

$$\zeta'(0) = \lim_{s \rightarrow 0} \frac{d}{ds} \sum_{k=1}^{\infty} \frac{1}{k^s} = -\lim_{s \rightarrow 0} \sum_{k=1}^{\infty} \frac{\ln k}{k^s} = -\frac{1}{2} \ln 2\pi \quad (35)$$

$$\prod_{k=1}^{\infty} k = \lim_{s \rightarrow 0} \prod_{k=1}^{\infty} e^{\frac{\ln k}{k^s}} = e^{-\zeta'(0)} = (2\pi)^{\frac{1}{2}} \quad (36)$$

A consequences of (36) which we shall use is

$$\prod_{k=1}^{\infty} (2\pi k) = 1 \quad (37)$$

B World line path integral

In this appendix, we will demonstrate that the one-loop vacuum energy of a scalar particle is given exactly by the world-line path integral when we define the various

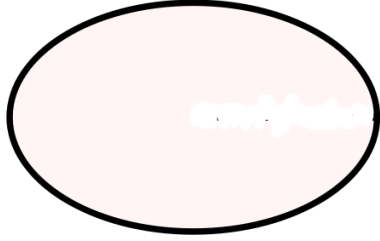


Figure 1: The Feynman diagram which must be computed to find the vacuum energy of a scalar field.

infinite products and sums which occur in the latter using zeta function regularization. The result will be equation (50).

We begin with the usual expression for the vacuum energy density of a complex scalar field in Euclidean space

$$\Gamma = \int \frac{d^4 p}{(2\pi)^4} \ln(p^2 + m^2) \quad (38)$$

which is represented by the Feynman diagram in figure 1. Recall that a real scalar field would be the same expression with a factor of $1/2$ in front. At this level, a complex scalar field is simply two real scalar fields which have twice as much vacuum energy. Here, the four-momentum is Euclidean and a space-time volume factor has been removed so that the result of doing the integral is the energy density. We will reorganize this integral in order to represent it as a world-line path integral. The singular integrals which we encounter will be defined by zeta function regularization. To proceed, we first introduce a Schwinger parameter, T ,

$$\Gamma = \int_0^\infty \frac{dT}{T} \int \frac{d^4 p}{(2\pi)^4} e^{-T(p^2 + m^2)} \quad (39)$$

We note that, in four dimensions, this integral contains an ultraviolet divergence, coming from the $T \sim 0$ integration region. This divergence must be regulated in order to obtain a sensible definition of the vacuum energy. We could regulate this expression by defining it as the limit

$$\ln(p^2 + m^2) = \lim_{\kappa \rightarrow 0} \frac{d}{d\kappa} (p^2 + m^2)^\kappa = \lim_{\kappa \rightarrow 0} \frac{d}{d\kappa} \frac{1}{\Gamma[\kappa]} \int \frac{dT}{T^{1+\kappa}} e^{-T(p^2 + m^2)} \quad (40)$$

The appropriate quantity to study would then be $\int \frac{dT}{T^{1+\kappa}} e^{-T(p^2 + m^2)}$ with the exponent of T shifted by κ which we could always take as negative with large enough magnitude that the integrals to be done converge, and then define the quantity in the region near $\kappa = 0$ by analytic continuation. In the following, we shall assume that this regulator is implicitly there, if needed but we will stick with expression (39) as the writing will be slightly simpler.

Now, consider the functional integral

$$\int [dx^\mu(\tau)] e^{i \int_0^1 d\tau p_\mu(\tau) \frac{d}{d\tau} x^\mu(\tau)} \quad (41)$$

where both $x(\tau), p(\tau)$ have periodic boundary conditions, $x^\mu(\tau + 1) = x^\mu(\tau)$ and $p_\mu(\tau + 1) = p_\mu(\tau)$. To define this functional integral, we use the expansions of the integration variables in a discrete orthonormal complete set of periodic functions,

$$p_\mu(\tau) = p_\mu + \sum_{k=1}^{\infty} \left[p_{\mu k} \sqrt{2} \sin(2\pi k\tau) + \tilde{p}_{\mu k} \sqrt{2} \cos(2\pi k\tau) \right] \quad (42)$$

and

$$x^\mu(\tau) = x^\mu + \sum_{k=1}^{\infty} \left[x_k^\mu \sqrt{2} \sin(2\pi k\tau) + \tilde{x}_k^\mu \sqrt{2} \cos(2\pi k\tau) \right] \quad (43)$$

The complete set of orthonormal periodic functions is $(1, \sqrt{2} \sin(2\pi k\tau), \sqrt{2} \cos(2\pi k\tau))$, where the normalization is the square-integral over the interval $\tau \in [0, 1]$. The functional integration measure is then defined as the ordinary Riemann integral over each of an infinite number of real variables,

$$[dx^\mu(\tau)] \equiv dx^\mu \prod_{k=1}^{\infty} dx_k^\mu d\tilde{x}_k^\mu, \quad [dp_\mu(\tau)] \equiv dp_\mu \prod_{k=1}^{\infty} dp_{\mu k} d\tilde{p}_{\mu k} \quad (44)$$

Now, with these definitions,

$$\begin{aligned} \int [dx^\mu(\tau)] \exp(i \int_0^1 d\tau p_\mu(\tau) \frac{d}{d\tau} x^\mu(\tau)) &= \int dx^\mu \prod_{k=1}^{\infty} dx_k^\mu d\tilde{x}_k^\mu \exp \left(i \sum_{k=1}^{\infty} (2\pi k i) [p_{\mu k} \tilde{x}_k^\mu - \tilde{p}_{\mu k} x_k^\mu] \right) \\ &= V \prod_{\mu=1}^4 \prod_{k=1}^{\infty} (2\pi) \delta(2\pi k p_{\mu k}) \cdot (2\pi) \delta(2\pi k \tilde{p}_{\mu k}) = V (2\pi)^{8\zeta(0)} \left(\frac{1}{\prod_{k=1}^{\infty} (2\pi k)} \right)^8 \prod_{k=1}^{\infty} \delta(p_{\mu k}) \delta(\tilde{p}_{\mu k}) \\ &= V \frac{1}{(2\pi)^4} \prod_{k=1}^{\infty} \delta(p_{\mu k}) \delta(\tilde{p}_{\mu k}) \end{aligned} \quad (45)$$

where $V \equiv \int dx^\mu$ is the (infinite) space-time volume arising from the integral over the constant mode x^μ and the fact that it does not appear in the integrand. We have used zeta function regularization to define the infinite products.

The identity

$$\int [dx^\mu(\tau)] \exp(i \int_0^1 d\tau p_\mu(\tau) \frac{d}{d\tau} x^\mu(\tau)) = V \frac{1}{(2\pi)^4} \prod_{k=1}^{\infty} \delta(p_{\mu k}) \delta(\tilde{p}_{\mu k}) \quad (46)$$

tells us that, if we first do the integral over $x^\mu(\tau)$ in the following path functional integral

$$\int \frac{d^4 p}{(2\pi)^4} e^{-T(p^2 + m^2)} = \frac{1}{V} \int [dx^\mu(\tau)] [dp_\mu(\tau)] e^{\int_0^1 \left[i p_\mu(\tau) \frac{d}{d\tau} x^\mu(\tau) - T(p_\mu^2(\tau) + m^2) \right]} \quad (47)$$

it will generate a factor of V and delta functions for all of the nonzero modes of $p_\mu(\tau)$. The integrals over those nonzero modes can then be done, leaving the integral over

the constant mode in $p_\mu(\tau)$ which becomes the momentum that appears on the left-hand-side of the equation. Now, we reorganize the right-hand-side of (47) by doing the functional integral over $p_\mu(\tau)$.

$$\begin{aligned}
& \int [dp_\mu(\tau)] e^{\int_0^1 \left[ip_\mu(\tau) \frac{d}{d\tau} x^\mu(\tau) - T(p_\mu^2(\tau) + m^2) \right]} \\
&= \int dp_\mu \prod_{k=1}^{\infty} dp_{\mu k} d\tilde{p}_{\nu k} e^{\sum_{k=1}^{\infty} [(2\pi k i) [p_{\mu k} \tilde{x}_k^\mu - \tilde{p}_{\mu k} x_k^\mu] - T(p_{\mu k}^2 + \tilde{p}_{\mu k}^2)] - T p^2} \\
&= \sqrt{\frac{\pi}{T} \left(\prod_{k=1}^{\infty} \frac{\pi}{T} \right)^2} e^{-\frac{1}{4T} \sum_{k=1}^{\infty} [(2\pi k x_k^\mu)^2 + (2\pi k \tilde{x}_k^\mu)^2]} = e^{-\frac{1}{4T} \int_0^1 d\tau \left(\frac{d}{d\tau} x^\mu(\tau) \right)^2} \quad (48)
\end{aligned}$$

where we have used the fact that the pre-factor is $\sqrt{\frac{\pi}{T} \left(\prod_{k=1}^{\infty} \frac{\pi}{T} \right)^2} = \left(\frac{\pi}{T} \right)^{\zeta(0)+1/2} = 1$. The result is then

$$\int \frac{d^4 p}{(2\pi)^4} e^{-T(p^2 + m^2)} = \frac{1}{V} \int [dx^\mu(\tau)] e^{\int_0^1 \left[-\frac{1}{4T} \dot{x}^\mu(\tau)^2 - T m^2 \right]} \quad (49)$$

where the dot denotes τ -derivative and we have the path integral formula for the vacuum energy density of a complex scalar field

$$\boxed{\Gamma = \frac{1}{V} \int_0^\infty \frac{dT}{T} \int [dx^\mu(\tau)] e^{-\int_0^1 d\tau \left[\frac{1}{4T} \dot{x}^\mu(\tau)^2 + T m^2 \right]}, \quad x_\mu(\tau + 1) = x_\mu(\tau)} \quad (50)$$

What we have shown is that, if one uses zeta function regularization, equation (50) is an identity.

We can check this identity by doing the path integral directly. We begin by doing the functional integral in the world-line expression (50) that we have derived. The integral over the constant mode of $x^\mu(\tau)$ gives a volume factor which cancels the factor of $1/V$. We can then use the rules for doing Gaussian integrals to do the quadratic functional integral over nonzero modes of $x^\mu(\tau)$. The result is

$$\begin{aligned}
\Gamma &= \int_0^\infty \frac{dT}{T} e^{-T m^2} \left[\prod_{k=1}^{\infty} \frac{2\pi \cdot 2T}{(2\pi k)^2} \right]^4 = \int_0^\infty \frac{dT}{T} e^{-T m^2} [4\pi T]^{4\zeta(0)} \\
&= \int_0^\infty \frac{dT}{T} e^{-T m^2} \frac{1}{(4\pi T)^2} \quad (51)
\end{aligned}$$

Again, we have used zeta function regularization to define the infinite products. The result is identical to what is obtained by integrating (39) over p .

Now, consider the vacuum energy of a charged scalar particle coupled to an electromagnetic field whose vector potential is $A_\mu(x)$. The coupling is implemented in the Euclidean functional integral by including the Bohm-Aharonov phase factor, to obtain

$$\Gamma = \frac{1}{V} \int_0^\infty \frac{dT}{T} \int_0^1 [dx^\mu(\tau)] e^{-\int_0^1 d\tau \left[\frac{1}{4T} \dot{x}^\mu(\tau)^2 + T m^2 \right] + i \oint d\tau \dot{x}^\mu(\tau) A_\mu(x(\tau))} \quad (52)$$

In particular, if we couple to an external constant electric field E , the gauge field could be taken as

$$A_\mu(x) = (0, iEx^1(\tau), 0, 0) \quad (53)$$

A physical electric field obtains a factor of i in Euclidean space. Now the path integral would be the Gaussian integral

$$\Gamma = \frac{1}{V} \int_0^\infty \frac{dT}{T} \int [dx^\mu(\tau)] e^{-\int_0^1 [\frac{1}{4T} \dot{x}^\mu(\tau) \dot{x}^\mu(\tau) + Ex^1(\tau) \dot{x}^2(\tau) + Tm^2]} \quad (54)$$

This integral should yield the vacuum energy of a charged scalar field which is coupled to an external electric field.

C A simple example

In this appendix, we shall test the saddle point approximation for the case of an integral which is similar to, but is much simpler than the worldline path integral (54) that we discussed in the Appendix above. The idea is to compute

$$I = \Im \int_0^\infty \frac{dT}{T} \int d^2x e^{-m^2 T - (T - T_0) \vec{x}^2} \quad (55)$$

The integral over \vec{x} is well-defined when $T > T_0$ and it diverges when $T < T_0$. We shall define the integral by assuming that T is in the region where the \vec{x} -integration is well-defined, doing the integral and continuing the result to the entire T -plane. What we then find is a pole at $T = T_0$. This analytic continuation results in the integral having an imaginary part due to the pole, where the integration contour must be defined using an “ $i\epsilon$ prescription”. The gaussian integral over \vec{x} produces

$$I = \Im \pi \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{T}} \frac{1}{T - T_0} \quad (56)$$

The prescription is to replace this integral by

$$I = \Im \pi \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{T}} \frac{1}{T - T_0 + i\epsilon} \quad (57)$$

and to use the formula

$$\frac{1}{T - T_0 + i\epsilon} = \frac{\mathcal{P}}{T - T_0} - i\pi\delta(T - T_0) \quad (58)$$

so that

$$I = -\frac{\pi^2}{T_0} e^{-\frac{m^2}{T_0}} \quad (59)$$

Now, let us do the integral using the saddle point technique. We consider the action

$$S = \frac{m^2}{T} + (T - T_0) \vec{x}^2 \quad (60)$$

The classical equations of motion are

$$\frac{m^2}{T^2} = \vec{x}^2 \quad , \quad (T - T_0) \vec{x} = 0 \quad (61)$$

These have a solution where

$$\vec{x} = \frac{m}{T_0} \hat{n} \quad , \quad T = T_0 \quad (62)$$

with \hat{n} an arbitrary unit vector. The action evaluated on this solution is

$$S_{\text{cl}} = \frac{m^2}{T_0} \quad (63)$$

Then, we consider fluctuations,

$$\vec{x} = \frac{m}{T_0} \hat{n} + \delta \vec{x} \quad , \quad T = T_0 + \delta T \quad (64)$$

The quadratic approximation to the action is

$$S = \frac{m^2}{T_0} + \frac{2m^2}{T_0^3} \frac{\delta T^2}{2} + \frac{2m}{T_0} \delta T \hat{n} \cdot \delta x + \dots + \dots \quad (65)$$

The Gaussian integral over δT produces a measure factor

$$\sqrt{\frac{\pi T_0^3}{m^2}} \quad (66)$$

and the remaining action becomes

$$S_1 = \frac{m^2}{T_0} - 2T_0 \frac{(\hat{n} \cdot \delta \vec{x})^2}{2} + \dots \quad (67)$$

Now, the degree of freedom $\hat{n} \cdot \delta x$ is tachyonic and its integral produces the measure factor

$$\sqrt{-\frac{\pi}{T_0}} \quad (68)$$

As well, there is an integral over the other component, $\hat{n} \times \delta \vec{x}$ which appears to be divergent. This apparent divergence is due to a symmetry which must be handled by the collective coordinate technique. For this purpose, we introduce the identity

$$1 = \frac{1}{2} \int_0^{2\pi} d\theta \delta(\hat{n}_\theta \times \vec{x}) \left| \frac{d}{d\theta} [\hat{n}_\theta \times \vec{x}] \right| \quad (69)$$

into the original integral.² Then, by changing the integration variable in the integrand $\delta x \rightarrow \delta x_\theta$ the entire integral becomes independent of θ .

The Jacobian becomes

$$\left. \frac{d}{d\theta} \hat{n}_\theta \times \vec{x}_{\theta'} \right|_{\theta'=\theta} = \hat{n} \cdot \left(\frac{m}{T_0} \hat{n} + \delta x \right) = \frac{m}{T_0} + \dots \quad (70)$$

and the integration over θ produces an additional factor of 2π . Gathering the measure factors, we find the result

$$I = \Im \frac{e^{-\frac{m^2}{T_0}}}{T_0} \cdot \sqrt{\frac{\pi T_0^3}{m^2}} \cdot \sqrt{\frac{-\pi}{T_0}} \cdot \frac{1}{2} \cdot \frac{m}{T_0} \cdot 2\pi = \pm \frac{e^{-\frac{m^2}{T_0}}}{T_0} \cdot \pi^2 \quad (71)$$

This produces the result of the first, exact evaluation but with a sign ambiguity. The correct sign must be chosen to reproduce the exact result.

We have found the exact result for the imaginary part of the integral by computing the integral in the “instanton” sector to the leading and next-to-leading order in a saddle point approximation. This suggests that the higher order corrections must all vanish. We leave a proof of this as an exercise for the reader.

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²The factor of 2 cancels a Gribov copy. If $\hat{n} = (1, 0)$, then $n_\theta = (\cos \theta, \sin \theta)$ and if $\vec{x} = x(\cos \phi, \sin \phi)$, $\hat{n}_\theta \times \vec{x} = \sin(\theta - \phi)x$ which has two zeros in the range $0 \leq \theta < 2\pi$. The factor of $1/2$ cancels this multiplicity of zeros of the constraint.

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